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Critical quantum spin chains as conformal field theories

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Abstract : We present a series of one-dimensional quantum Hamiltonians which, at their critical points, realise the minimal unitary series of conformal field theories with central charge less than 1 The models consist of ferromagnetically coupled SU(2) spins in a transverse magnetic field We show how the infinite spin (free boson) limit can be obtained by using the Holstein-Primakoff transformation. The analysis can be generalised to SU(3) quantum chains which realise a different series of conformal field theories at criticality

Keywords : Quantum spin chains, conformal field theories, Landau-Ginzburg theory, multicritical points, large S limit, Holstein-Primakoff transformation

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1. Introduction

At critical points describing second order phase transitions, long-distance phenomena in statistical mechanics are not governed by any intrinsic length scales. For instance, the two-point correlations at points separated by distances much larger than the microscopic lengths typically fall off as a power of the separation, and not as an exponential (as they would away from criticality). Statistical mechanics models in d spatial dimensions are generally related to quantum field theories in the same number of space-time dimensions [1]. In the language of field theory, we say that a critical theory is invariant under scale transformations. In fact, it turns out that the theory is invariant under the full conformal group (This group keeps angles between two infinitesimal vectors at each point in space invariant). This larger invariance is specially powerful in two dimensions where the conformal group is infinite dimensional [2]. If we think of the two dimensions as forming a complex plane, then the most general conformal transformation is given by $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$, where f is any analytic function of z (a function which can be expressed as a Laurent series in z). The symmetry group of a conformally invariant field theory (CFT) is described by a Virasaro algebra which is characterised by a real number c called the central charge. Soon after the introduction of CFT, statistical mechanics models were indentified which can realise such theories at criticality [3-9]. A very large number of exactly solvable models are now known to provide examples of CFT [10-16].

A class of CFT can be expressed as affine Kac-Moody algebras (which are extensions of the Virasaro algebra) based on classical Lie groups [17]. For exmple, $SU(2)_k$ denotes a level K affine algebra based on SU(2) where K is a positive integer. This has $c = \frac{3K}{K+2}$. As critical statistical mechanics models, these are realised by quantum spin chains with certain isotropic polynomial interactions between nearest neighbour spins, with $K = 2S$ for the spin-S model [14-16]. Similarly, the affine algebra $SU(3)_k$ has $c = \frac{8K}{K+3}$ (The central charge of G_k is $\frac{Kd_G}{K + c_A(G)}$, where d_G is the number of generators of the Lie algebra G and $C_A(G)$ is the value of the quadratic Casimir in its adjoint representation). Other CFT can be obtained from the above models by coset constructions [18]. Thus, the minimal unitary series of CFT with

$$c = 1 - \frac{6}{(K+2)(K+3)} \tag{1}$$

can be written as $\frac{SU(2)_k \times SU(2)_1}{SU(2)_{k+1}}$ (The value of c is simply obtained by adding up the values for the algebras in the numerator and in the denominator and subtracting the latter from the former). The series in (1) is realised by the models in Refs. [8] and [9] where there is an integer variable which sits on each site of a square lattice and takes vaules from 1 to $K + 2$. Integers on neighbouring sites must differ by one. The states on the lattice links can therefore take $K + 1$ different values. These states will be identified with spins in our models. For the cosets $\frac{SU(3)_k \times SU(3)_1}{SU(3)_{k+1}}$, the central charge is

$$c = 2\left[1 - \frac{12}{(K+3)(K+4)}\right] \tag{2}$$

These can occur in statistical models where two integer variables sit on each lattice site [10-13]. The states then correspond to the points in the weight diagrams of various representations of SU(3) [19].

In spite of the successes of the exactly solvable statistical models in providing concrete examples of CFT, it is instructive to study the subject from other points of view. Zamolodchikov's Landau-Ginzburg (LGZ) theory is one example of an alternative approach [20]. It describes the minimal series in (1) in terms of the $(K + 1)$-fold multicritical point of an effective Lagrangian with a single scalar field

$$\pounds = \frac{1}{2} (\partial_\mu \phi)^2 + g\, \phi^{2(K+1)} \tag{3}$$

Here ϕ is a function of two coordinates t and x. This theory has a multicritical point where $K + 1$ phases have merged and become indistinguishable from each other (Below the critical temperature, the potential $V(\phi)$ has $K + 1$ distinct minima with the same energy. These correspond to that many equilibrium phases with the same free energy. On approaching the critical temperature, the locations of all the minima move to $\phi = 0$): In the limit $K \to \infty$, the multicritical theory describes a free massless boson since the potential vanishes in the neighbourhood of $\phi = 0$. Such a boson constitutes a CFT with $c = 1$ which is also the infinite K limit of (1).

The facts that the series in (1) is given by cosets of the group SU (2) which has rank one, and that the LGZ description of it in (3) contains one scalar field are related.. Further, the LGZ potential $V(\phi) = g\,\phi^{2(K+1)}$ at the critical point has the Weyl symmetry of SU (2) [19]. Denote the three generators of SU (2) by (S_+, S_-, S_z) where S_z forms the Cartan subalgebra and $S_\pm$ are the ladder operators. The Weyl group has two elements. The non-trivial element reflects (S_+, S_-, S_z) to $(S_-, S_+, -S_z)$. We identify the expectation values of S_z in the various phases with the locations of the minima of $V(\phi)$ below the critical temperature. Under the Weyl reflection $\phi \rightarrow -\phi$, $V(\phi)$ and $(\partial_x \phi)^2$ in (3) are both invariant. In Section 2, the kinetic term $(\partial_t \phi)^2$ will be identified with $S_x = S_+ + S_-$. The sum of the ladder operators is certainly Weyl symmetric.

For the SU (3) coset models in (2), the LGZ description uses two scalar fields ϕ_1 and ϕ_2. SU (3) has two generators S_1 and S_2 in its Cartan subalgebra, and three pairs of ladder operators E_{a+} and their hermitian conjugates E_{a-}, where $a = 1$, 2 and 3. The expectation values of S_1 and S_2 in the various phases of a statistical model are denoted by ϕ_1 and ϕ_2 respectively. The LGZ Lagrangian is

$$\pounds = \frac{1}{2}\,(\partial_\mu \phi_1)^2 + \frac{1}{2}\,(\partial_\mu \phi_2)^2 - V(\phi_1, \phi_2) \tag{4}$$

The Weyl group now has six elements. The group is generated by $\frac{2\pi}{3}$ rotations in the (ϕ_1, ϕ_2) plane, and a reflection to $(-\phi_1, \phi_2)$. By Weyl symmetry, the potential can only be a function of $\alpha_\phi = \phi_1^2 + \phi_2^2$ and $\beta_\phi = \phi_2^3 - 3\phi_2\phi_1^2$. The term $(\partial_x\phi_1)^2 + (\partial_x\phi_2)^2$ is Weyl symmetric. It can again be shown that $(\partial_t\phi_1)^2 + (\partial_t\phi_2)^2$ arises from the generator

$$T = \sum_{a=1}^{3} (E_{a+} + E_{a-}) \tag{5}$$

which is also Weyl symmetric [21]. The first member ($K = 1$) of the SU (3) series has $c = 4/5$ and corresponds to the three-state Potts model. The LGZ Lagrangian for it has written long ago and it has the required Weyl symmetry [22].

In this work, we will express the two series in (1) and (2) in terms of quantum spin chain models. SU (2) is studied in detail in Section 2, while Section 3 discusses SU(3) very briefly. The spin chain models arise in the quantum Hamiltonian limit of the row-to-row transfer matrix $\hat{T}$ of appropriate two-dimensional statistical models [1, 23]. In this highly anisotropic limit, the couplings between the spins in one row and the next are taken to infinity while the couplings between spins in the same row are simultaneously taken to zero in proportion to a small parameter τ. As $\tau \rightarrow 0$, the transfer matrix takes the form $\hat{T} = \exp(\tau H_Q)$ where H_Q is the Hamiltonian of a quantum chain. The ground state (or states) of this model corresponds to the equilibrium phase (or phases) of the statistical model. Finally, in the limit that the spacing between neighbouring spins of the chain go to zero, H_Q becomes the Hamiltonian of a (1 + 1)-dimensional quantum field theory.

The one-dimensional quantum systems we will study are not exactly solvable, but they have the virtue of being amenable to well-known methods of perturbative and numerical analysis (We will exhibit the results of second order perturbation theory below). We have the additional motivations that CFT have not so far been extensively studied from this point of view, and that the connection to the LGZ description is more transparent for the quantum chains than for the exactly solvable statistical models. (The elegant mathematics used to solve the latter models sometimes tends to obscure the relatively simple physics). We will identify the different spin chain models with the appropriate CFT using symmetry and universality arguments. However, we will not attempt to calculate the central charges or the scaling dimensions of the various operators present in our models.

A key role in our analysis is played by the Holstein-Primakoff (HP) transformation from the generators of SU(N) to bosonic creation and annihilation operators. This is well-known and widely used for SU(2) spins [24-26], and it can be generalised to the symmetric representations of SU(3) [21]. HP is very useful for studying the infinite K (free boson) limit of the SU(N) cosets in eqs. (1) and (2).

2. SU(2) quantum spin chains

At a second order critical point, the quantum models have a vanishing energy gap. The spectra of low energy excitations (with wavelengths much larger than the lattice spacing) consists of one or more relativistic massless dispersion relations $\omega(k) = vk$, where v is the "velocity of light" and k is the momentum. In this way, a CFT is recovered. Consider, for example, the quantum model which is related to the two-dimensional Ising model with nearest neighbour ferromagnetic interactions. The hamiltonian for a chain with N sites is

$$H_Q = \sum_{i=1}^{N} \left[\frac{1}{2} (S_{iz} - S_{i+1,z})^2 + \gamma S_{ix} \right] \qquad (6)$$

A spin-1/2 object sits at each site. The operators S_{ia} are given by the three Pauli matrices $\sigma_{ia}/2$. The transverse magnetic field γ plays the role of temperature in the corresponding statistical model. For small fields $0 \le \gamma < \gamma^*$ (where $\gamma^* = 1/2$), the ground state is doubly degenerate in the thermodynamic limit $N \to \infty$. Both states are ordered and have non-zero expectation values $<S_z> = \frac{1}{N} \sum_i <S_{iz}>$. The values in the two states have opposite signs by Weyl symmetry. For $\gamma > \gamma^*$, there is an unique disordered ground state with $<S_z> = 0$. In both these regimes, the low energy spectrum has a gap and correlation functions decay exponentially at large distances. As a relativistic field theory, one has a free massive Majorana fermion [27, 28]. Exactly at the critical point γ^*, the energy gap vanishes and the long distance correlations decay algebraically. We then get a massless Majorana fermion which constitutes a CFT with $c = 1/2$. This is the first member ($K = 1$) of the SU(2) coset series in (1).

We will generalise (6) to higher spins and argue that these models have a multicritical point realising a CFT with c given in (1). The integer K is related to the spin by $K = 2S$. Our models are perhaps the simplest possible ones which are Weyl symmetric and have a critical point where $K + 1$ phases simultaneously become indistinguishable. In addition to the terms in (6), the models contain an on-site interaction given by a finite polynomial in S_{iz}^2. The degree of the polynomial is $[S]$, the largest integer less than or equal to S. The Hamiltonian is

$$H_Q = \sum_i \left[\frac{1}{2}(S_{iz} - S_{i+1,z})^2 + \gamma S_{ix} + \sum_{n=1}^{[S]} a_{2n} S_{iz}^{2n} \right] \tag{7}$$

Consider the phase diagram of (7) in the $[S] + 1$ dimensional parameter space (γ, $\vec{a}_{2n}$). At the origin (0, $\vec{0}$), the ground state is ($K + 1$)-fold degenerate. In each state, the spins at all sites have the same value of $S_z = m$ where $m = -S, -S+1, \ldots, S$ distinguishes between the various phases (We will use the notation $<S_z> = m\ (\gamma)$ and $m = m\ (0)$ below). As γ is increased from zero, the coefficients a_{2n} must be simultaneously tuned in order to maintain the same degree of degeneracy for the ground state. Further, in any state m, the spins will start fluctuating to values of S_z different from m and $< S_z >$ will begin to approach zero. When γ reaches a critical value γ^*, and the a_{2n}'s correspondingly go to a_{2n}^*, the $K + 1$ states become indistinguishable and one has an unique disordered ground state with $<S_z> = 0$. The number $[S]$ of the a_{2n}'s is precisely enough so as to be able to tune $K + 1$ states to the same energy. We therefore have a line A parametrised by γ which runs from the origin (0,0) to the multicritical point $P = (\gamma^*, \vec{a}_{2n}^{\,*})$.

At the point P, one expects a CFT with c given by (1). This is because our models have the same (Weyl) symmetry as the ones solved exactly in Refs. [8] and [9], and have the same degree of multicriticality $K + 1$. By universality, they must both describe the same CFT at that critical point (For S = 1/2 and 1, we know from earlier works that CFT with $c = 1/2$ and 7/10 are obtained [3, 27-29].

We can find an equation for the line A by a perturbative expansion in small γ (This is analogous to a low temperature expansion in statistical mechanics). Along that line in the phase diagram, we study how the expectation value $m\ (\gamma)$ in a given phase m approaches zero. By applying a simple ratio test, we estimate the location of the critical point P (For illustrative purposes, we will present the phase diagrams for spin-1 and spin-3/2). Finally, we argue that it is consistent for the $S \to \infty$ limit to reduce to a free massless boson at criticality.

At the beginning the line A where $\gamma = 0$, all the $K + 1$ states have zero energy. In any state m, excitations consist of one or more spins differing from m. Using Rayleigh-Schrödinger perturbation theory, we determine how the ground state energies $E\ (m, \gamma, \vec{a}_{2n})$ change with increasing γ, and how the a_{2n}'s must be adjusted so that $E(m, \gamma, \vec{a}_{2n})$ may continue to remain independent of m. We take S to be arbitrary in our calculations and keep all the a_{2n}'s from $n = 1$ to ∞. For any particular S, we can then truncate to the first $[S]$ a_{2n}'s by using the fact that S_z^{2n} for $n > [S]$ can be expressed in terms of the first $[S]$ powers of S_z^2 and the unit matrix I.

To order γ^4, we discover that only a_2 and a_4 need to be changed from zero, that is,

$$a_2 = -\frac{1}{2}\gamma^2 + \left[\frac{5}{48} S(S+1) - \frac{25}{96}\right]\gamma^4$$

$$a = -\frac{5}{96}\gamma^4 \qquad (8)$$

It can be shown in general that the series expansion for a_{2n} begins at order γ^{2n}. This will prove to be important later.

The line $A = (\gamma, \bar{a}_{2n}(\gamma))$ is actually a first order transition line on which $K + 1$ different phases coexist. Take $S = 1$ for example. The Hamiltonian in (7) is a particular case of the Blume-Emery-Griffiths model [29], which is known to have a tricritical point (realising a CFT with $c = 7/10$). Figure 1 exhibits the phase diagram. In the ordered region O lying below the line A, the phases $m = 1$ and -1 coexist with $<S_z> > 0$ and < 0

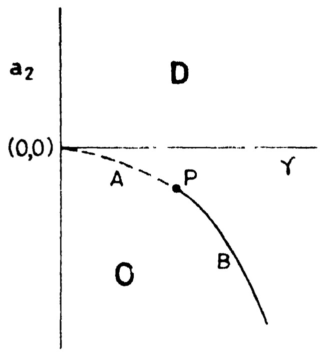

Figure 1. Phase diagram of the spin-1 model. The dashed and solid lines A and B indicate lines of first order and second order transitions respectively. O and D denote ordered and disordered regimes. P is a tricritical point.

respectively. Above A, the disordered ground state D has $<S_z> = 0$. As A approaches the tricritical point P, the values of $<S_z>$ in the phases $m = \pm 1$ go to zero. The solid line B in the figure is a line of second order phase transition points of the usual Ising type (a CFT with $c = 1/2$). Ref. [29] numerically estimates the coordinates of P to be $(0.416, -0.090)$ which is consistent with eq. (8). As another example, the $S = 3/2$ phase diagram is shown in Figure 2. This has two ordered phases O_1 and O_2. In O_1, the phases $m = \pm 3/2$ coexist while in O_2, $m = \pm 1/2$ coexist. At the point P, all four phases coalesce to produce a CFT with $c = 4/5$.

The phase diagrams become increasingly complicated as S increases. The multicritical point P always lies at an intersection of the first order line A and several second order surfaces. Although the complete phase diagram is hard to visualise as $S \to \infty$, we will see later that the model becomes quite simple exactly at P.

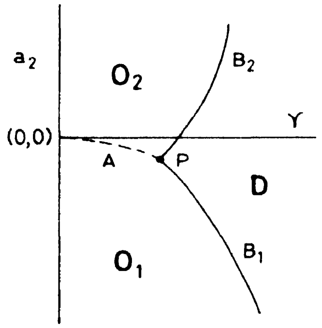

Figure 2. Phase diagram for spin-3/2. There are two ordered regimes O_1 and O_2, and two second order transition lines B_1 and B_2. P is a multicritical point.

We now consider the expectation value $<S_z>$. One could study this in any one of the $K+1$ phases distinguished by m. Following Refs. [1] and [23] and using the expansion in (8), we find $m(\gamma)$ to order γ^4.

$$\frac{m(\gamma)}{m(o)} = 1 - \frac{\gamma^2}{2} - \gamma^4 \left[\frac{17}{16} S(S+1) - \frac{17}{16} m^2 + \frac{11}{32} \right] \tag{9}$$

We now apply a ratio test to determine the critical point γ^* and an exponent β. We assume that $m(\gamma)/m(0)$ fits the formula $\left(1 - \frac{\gamma^2}{\gamma^{*2}}\right)^{\beta}$ and expand the latter to order γ^4. A comparison with (9) yields two equations for γ^{*2} and β. These give

$$\frac{1}{\gamma^{*2}} = \frac{17}{4} \left[S(S+1) - m^2 \right] + \frac{15}{8} \tag{10}$$

and $\beta = \gamma^{*2}/2$ (Note that for $S = 1/2$ and $m = \pm 1/2$, these values of γ^* and β agree with the exact results [1, 23]). Unfortunately, the estimate for γ^* in (10) depends on the variable m. Since we are eventually interested in the infinite S limit and want results which are independent of S in that limit, we should consider values of m much smaller than S. We thus get $\gamma^{*2} \sim \frac{4}{17} S^{-2}$. However, this value should not be taken too seriously because we only used two terms (the minimum number necessary) in order to apply the ratio test. The only information we can perhaps extract with some confidence from the above is that $\gamma^* \sim 1/S$ as $S \to \infty$. Since a_{2n} starts with γ^{2n}, it is then reasonable to suppose that $a_{2n}^* \sim S^{-2n}$. It is, of course, necessary to go to higher orders in perturbation theory to check if these conjectures are true.

Now we do a spin wave analysis at large S [24-26]. At $S \to \infty$, the configuration of spins can be viewed classically to lowest order in $1/S$. Since S is absent in (7), the ground state must have all the spins lying in the $(\hat{x}, \hat{z})$ plane. At $\gamma = 0$, the spins all point in the same direction and the ground state energy is zero regardless of that direction. As the magnetic field increases, the spins start tilting towards the negative $\hat{x}$-direction. At γ^*, they all have the classical values $(S_x, S_y, S_z) = (-S, 0, 0)$.

To the next order in $1/S$, the operators S_y and S_z undergo quantum fluctuations about zero. To quantify this, we perform the HP transformation at each site.

$$\begin{aligned} S_y + i\, S_z &= a^+ \,(2S - N)^{1/2} \\ S_y + i\, S_z &= (2S - N)^{1/2} a \\ S_x &= -S + N \end{aligned} \tag{11}$$

where $[a, a^+] = 1$ and N is the number operator $a^+ a$. We expand (11) for large S and keep only the lowest order terms. On defining the canonically conjugate variables $q = \frac{i}{\sqrt{2}}(a - a^+)$ and $p = \frac{1}{\sqrt{2}}(a + a^+)$, we find that

$$\begin{aligned} q_i &= \frac{S_{iz}}{\sqrt{S}} \\ p_i &= \frac{S_y}{\sqrt{S}} \\ S_{ix} &= -S - \frac{1}{2} + \frac{1}{2}\,(q_i^2 + p_i^2) \end{aligned} \tag{12}$$

where $[q_i, p_j] = i\,\delta_{ij}$.

For $S \to \infty$, let us take $\gamma^* = \alpha S^{-1}$ where α is a positive constant of order 1, and $a_{2n} = \alpha_{2n} S^{-2n}$. Then (7) reduces to

$$H_Q = \frac{1}{2} \sum_i \left[\frac{\alpha}{S}\, p_i^2 + S\,(q_i - q_{i+1})^2 \right] \tag{13}$$

where higher order terms in $1/S$ and a constant have been dropped. On Fourier transforming, we obtain the spin wave spectrum

$$\omega(k) = 2\alpha^{1/2} \left| \sin \frac{k}{2} \right| \tag{14}$$

where the momentum k lies in the range $[-\pi, \pi]$. In the continuum limit $k \to 0$, this is the relativistic dispersion for a free massless boson with "velocity" $v = \alpha^{1/2}$. If we identify $\phi(t, x)$ with $S_{iz} = q_i \sqrt{S}$, we see that $(S_{iz} - S_{i+1,z})$ in (7) becomes $(\partial_x \phi)^2$ in the continuum limit. The kinetic term $(\partial_t \phi)^2 = \alpha p_i^2 S^{-1}$ comes from the magnetic field operator γS_{ix}.

3. SU(3) quantum chains

We now briefly discuss how the spin chain Hamiltonian can be set up for the case of SU(3). We will consider the symmetric representations of SU(3) since we know the HP transformation for only these at the moment. Such a representation is denoted by $(K, 0)$

where K is a positive integer, and its dimensionality is $d\,(K) = \frac{1}{2}\,(K+1)\,(K+2)$. In the weight diagram, these $d\,(K)$ points form an equilateral triangle of side $2K$. In Figure 3, we show the $K = 2$ representation for illustration. The two-way arrows marked 1, 2 and 3 denote the directions along which the ladder operators $E_{1\pm}$, $E_{2\pm}$ and $F_{3\pm}$ move the six points. The quadratic Casimir invariant is given by

$$C_k = S_1^{\,2} + S_2^{\,2} + 2\sum_a \{E_{a+}\,, E_{a-}\} = \frac{4}{3}\,K\,(K+3) \tag{15}$$

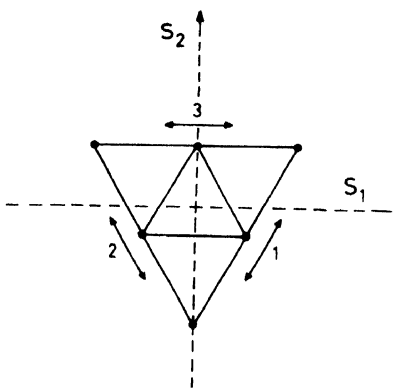

Figure 3. The 6 representation of SU(3). S_1 and S_2 generate the Cartan subalgebra. The arrows marked 1, 2 and 3 indicate the directions along which the three pairs of ladder operators E_a, act.

The obvious generalisation of Section 2 is to consider a chain with a spin lying in the $(K, 0)$ representation of SU(3) sitting at each site. As pointed out in Section 1, the operators T in (5), $\alpha_s = S_1^{\,2} + S_2^{\,2}$ and $\beta_s = S_2^{\,3} - 3S_2S_1^{\,2}$ are all Weyl invariant. So a candidate quantum Hamiltonian is

$$H_Q = \sum_i \left[\frac{1}{2}\,(S_{i\,1} - S_{i+1,\,1})^2 + \frac{1}{2}\,(S_{i\,2} - S_{i+1,\,2})^2 - \gamma T_i + P\,(\alpha_{is}, \beta_{is})\right] \tag{16}$$

where $P\,(\alpha_s, \beta_s)$ is a polynomial in α_s and β_s. As before, we can argue on grounds of symmetry and universality that a CFT with c given in (2) will emerge at the $d\,(K)$-fold multicritical point of the above model.

For $K = 1$ (the fundamental representation denoted by 3), α_s has the same value for each of the three possible states, as does β_s. Hence the polynomial $P\,(\alpha_s, \beta_s)$ can be dropped (we ignored a polynomial in $S_z^{\,2}$ in (6) for spin-1/2 for the same reason). The remaining pieces in H_Q precisely give the quantum Hamiltonian of the three-state Potts model [30]. At $\gamma = 0$, this has three phases denoted by three points in a plane (ϕ_1, ϕ_2). As γ increases to a critical value $\gamma^* = 2/3$, these points converge to the origin (0, 0) where a $c = 4/5$ CFT resides.

For any value of $K > 1$, the polynomial $P\,(\alpha_s, \beta_s)$ will have a finite degree. The degree must be such that there are a sufficient number of adjustable parameters to allow d (K) phases to simultaneously have the same ground state energy. Finally, one can again argue that a theory of two massless bosons results in the limit $K \to \infty$. For this purpose, the Holstein-Primakoff transformation from the eight generators of SU(3) to two independent pairs of bosonic creation and annihilation operators are required [21]. We then transform from these to the canonically conjugate operators q and p. Assuming that the polynomial interaction $P(\alpha_s, \beta_s)$ can be ignored as $K \to \infty$ (just as we did for SU(2) in Section 2), we eventually find that eq. (16) reduces to

$$H_Q = \sum_i \left[\frac{2K}{3} (q_{i\,1} - q_{i+1,1})^2 + \frac{2K}{3} (q_{i\,2} - q_{i+1,2})^2 + \frac{3\gamma^*}{2} (P_{i\,1}{}^2 + P_{i\,2}{}^2) \right] \tag{17}$$

In the continuum limit, relativistic dispersion relations for two massless bosons follow with the "velocity" $v = [4K\gamma^*]^{1/2}$. In appropriately scaled units, (17) leads to the Landau-Ginzburg hamiltonian

$$H = \frac{1}{2} \left[(\partial_t \phi_1)^2 + (\partial_t \phi_2)^2 + (\partial_x \phi_1)^2 + (\partial_x \phi_2)^2 \right] \tag{18}$$

where $q_{i\,1} = \phi_1(t, x)$ and $q_{i\,2} = \phi_2(t, x)$.

4. Outlook

The analysis in this work can be generalised in several ways. One can study the symmetric representations of the higher SU(N) almost immediately since the HP transformation can be easily extended to these. HP transformations for other representations of either SU(3) or the higher SU(N) are however not known to me.

One may then ask whether coset models of the form $\frac{G_k \times G_1}{G_{k+1}}$ can be studied in a similar way for other Lie groups G. Suppose that we are only interested in the infinite K limit (which may correspond to representations of G which grow large in a particular manner). Then the central charge tends to $c(G_1)$. If a LGZ description in terms of rank (G) free massless scalar fields is to be valid in that limit, one must have

$$c(G_1) = \text{rank}(G) \tag{19}$$

This equality holds only for the groups $A_N = \text{SU}(N+1)$, $D_N = \text{SO}(2N)$ and E_N (with $N =$ 6, 7 and 8). We may therefore consider generalising our analysis to various representations of these groups. For this purpose, it would certainly be useful to construct the appropriate HP transformations.

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